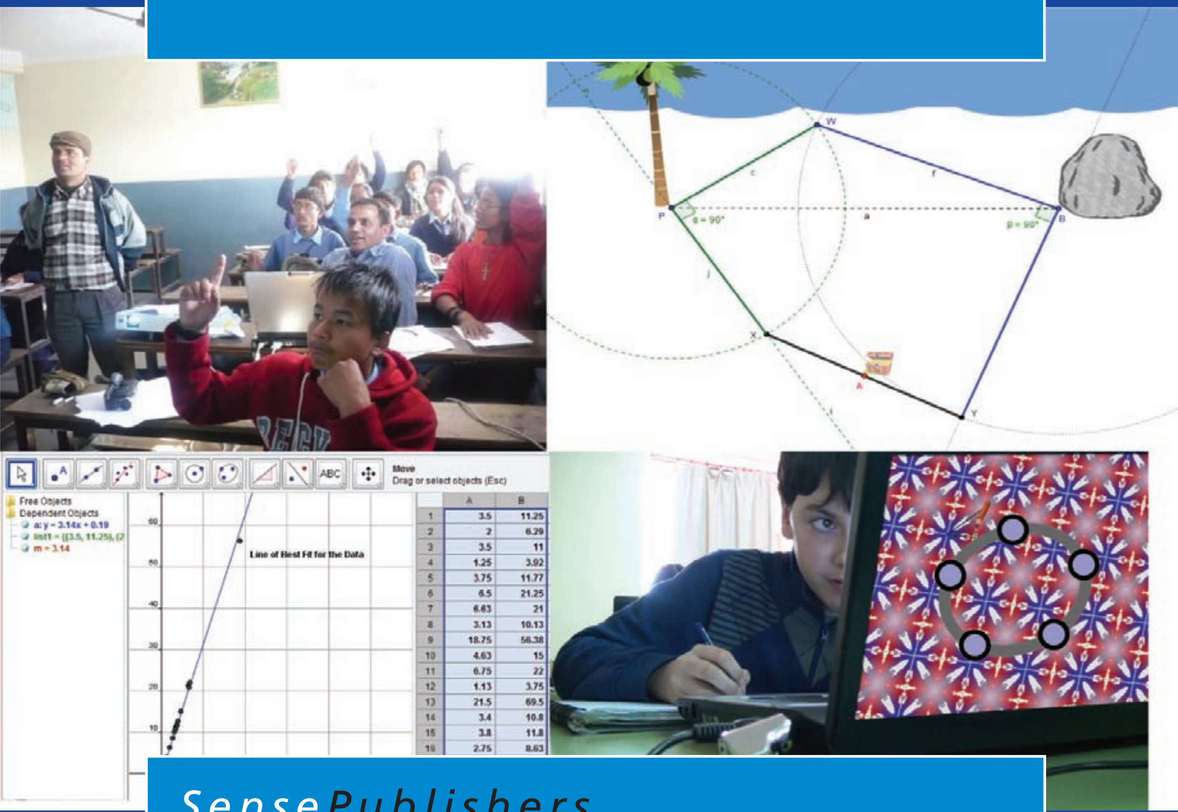
MODELING AND SIMULATIONS FOR LEARNING AND INSTRUCTION

Model-Centered Learning

Pathways to Mathematical Understanding Using GeoGebra

Lingguo Bu and Robert Schoen (Eds.)



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Model-Centered Learning

MODELING AND SIMULATIONS FOR LEARNING AND INSTRUCTION
Volume 6

Scope

Models and simulations have become part and parcel of advanced learning environments, performance technologies and knowledge management systems.This book series will address the nature and types of models and simulations from multiple perspectives and in a variety of contexts in order to provide a foundation for their effective integration into teaching and learning. While much has been written about models and simulations, little has been written about the underlying instructional design principles and the varieties of ways for effective use of models and simulations in learning and instruction. This book series will provide a practical guide for designing and using models and simulations to support learning and to enhance performance and it will provide a comprehensive framework for conducting research on educational uses of models and simulations. A unifying thread of this series is a view of models and simulations as learning and instructional objects. Conceptual and mathematical models and their uses will be described. Examples of different types of simulations, including discrete event and continuous process simulations, will be elaborated in various contexts. A rationale and methodology for the design of interactive models and simulations will be presented, along with a variety of uses ranging from assessment tools to simulation games. The key role of models and simulations in knowledge construction and representation will be described, and a rationale and strategy for their integration into knowledge management and performance support systems will provided.

Audience

The primary audience for this book series will be educators, developers and researchers involved in the design, implementation, use and evaluation of models and simulations to support learning and instruction. Instructors and students in educational technology, instructional research and technology-based learning will benefit from this series.

Model-Centered Learning

Pathways to Mathematical Understanding Using GeoGebra

Edited by

Lingguo Bu
Southern Illinois University, Carbondale, USA

Robert Schoen
Florida State University, Tallahassee, USA

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J. MICHAEL SPECTOR

FOREWORD

Today's students live in a world of ubiquitous technology. However, these technologies have not been adequately incorporated into learning and instruction. Mathematics education has evolved with the times and the available technologies. Calculators eventually made their way into schools. The battle goes on to persuade educators and parents and others that what was important was the ability to solve complex problems – not the ability to perform complex calculations on paper. Architects and engineers and scientists do not perform very many complex calculations on paper. They use sophisticated calculating devices. Within the context of authentic learning, it makes sense to make similar tools available to students.

Graphing calculators have been introduced in courses involving mathematics, engineering, and science. Why is that happening? It seems to be a natural evolution of the use of technology in education. Now that the burden of performing complex calculations has shifted to machines, the new burden of understanding the data that can be quickly calculated is receiving greater attention. Graphing calculators can help in understanding complex functions through a visual and dynamic representation of those functions.

Have calculators and graphing calculators had a significant impact on students' ability to understand relationships among variables and complex sets of data? It is probably the case that the impact has been less than advocates of these tools and technologies would like to believe. Given the lack of significant impact of previous innovative tools in mathematics education, what lessons can be learned that will contribute to future success with new tools?

I believe there are two important lessons to be learned. The first is that the proper preparation and training of teachers is critical to success when introducing new instructional approaches and methods, new learning materials, and innovative tools. The second is that new tools and technologies should be used in ways that support what is known about how people come to know and understand things. It is now widely accepted that people create internal representations to make sense of new experiences and puzzling phenomena. These internal representations or mental models are important for the development of critical reasoning skills required in many professional disciplines, including those involving mathematics. Using appropriate pedagogical methods and tools to support these internal representations is an important consideration for educators.

This volume is about GeoGebra, a new, cost-free, and very innovative technology that can be used to support the progressive development of mental models appropriate for solving complex problems involving mathematical relationships (see http://www.geogebra.org/cms/). GeoGebra is supported with many additional free resources, including lessons, examples, and activities that can be used to support the training of teachers in the integration of GeoGebra into curricula aligned with standards, goals and objectives. The topics herein range widely from using GeoGebra to model real-world problems and support problem solving, to provide visualizations and interactive illustrations, and to improve student motivation and cognitive development.

In short, this is an important book for mathematics educators. It is a must read for all secondary and post-secondary math teachers and teacher educators who are interested in the integration of GeoGebra or similar technologies in mathematics education. In addition, it is a valuable resource for all educators interested in promoting the development of critical reasoning skills.

J. Michael Spector
University of Georgia, USA

ACKNOWLEDGEMENTS

This book project was initially inspired by a series of discussions the editors had with Dr. J. Michael Spector in early 2008 about the roles of modeling and simulations in complex learning, including the potential applications and implications of GeoGebra in mathematics education. The editors are grateful to Dr. Spector for his unfailing support and ongoing encouragement in both theory and practice in relation to the book development. During the initial call for proposals and subsequent review and selection of manuscripts, Dr. Markus Hohenwarter and Dr. Zsolt Lavicza played a helpful role in contacting the international GeoGebra community to invite proposals and manuscripts. We extend our thanks to all the chapters' authors, who not only reviewed and contributed to their colleagues' work, but also made many constructive suggestions to the overall coverage of the book. We also would like to thank our graduate assistants at Southern Illinois University Carbondale for carefully reviewing and commenting on early drafts of the chapter manuscripts: Yazan Alghazo, Ashley Launius, Gilbert Kalonde, and Jia Yu. The Department of Curriculum and Instruction at Southern Illinois University Carbondale provided generous funding in support of travel and consulting related to the book project. We further wish to thank Peter de Liefde, at Sense Publishers, for his support and patience with us during the lengthy reviewing and editing process. Finally, we are indebted to our families, whose understanding and support helped us bring the book to fruition.

Lingguo Bu and Robert Schoen

LINGGUO BU AND ROBERT SCHOEN

GEOGEBRA FOR MODEL-CENTERED LEARNING IN MATHEMATICS EDUCATION

An Introduction

But common as it is, much of education clings too stubbornly to abstraction, without enough models to illustrate and enliven them. The cure for this on the learner's side is to call for more models. Learners need to recognize that they need models and can seek them out.

—Perkins (1986, p. 147)

Mental models serve a twofold epistemological function: They represent and also organize the subject's knowledge in such a way that even complex phenomena become plausible.

—Seel, Al-Diban, & Blumschein (2000, p. 130)

It makes no sense to seek a single best way to represent knowledge—because each particular form of expression also brings its own particular limitations.

—Minsky (2006, p. 296)

GeoGebra (http://www.geogebra.org) is a community-supported open-source mathematics learning environment that integrates multiple dynamic representations, various domains of mathematics, and a rich variety of computational utilities for modeling and simulations. Invented in the early 2000s, GeoGebra seeks to implement in a web-friendly manner the research-based findings related to mathematical understanding and proficiency as well as their implications for mathematics teaching and learning: A mathematically competent person can coordinate various representations of a mathematical idea in a dynamic way and further gain insight into the focal mathematical structure. By virtue of its friendly user interface and its web accessibility, GeoGebra has attracted tens of thousands of visitors across the world, including mathematicians, classroom math teachers, and mathematics educators. Through the online GeoGebra Wiki and global and local professional conferences, an international community of GeoGebra users has taken shape. This growing community is actively addressing traditional problems in mathematics education and developing new pedagogical interventions and theoretical perspectives on mathematics teaching and learning, while taking advantage of both technological and theoretical inventions. Meanwhile, in the fields of learning sciences and instructional design, researchers have highlighted the theoretical and practical implications of *mental models* and *conceptual models* in complex human learning (Milrad, Spector, & Davidsen, 2003; Seel, 2003). A

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 1–6.

model-centered framework on learning and instruction does not only help us understand the cognitive processes of mathematical sense-making and learning difficulties, but also lends itself to instructional design models that facilitates meaningful learning and understanding. Thus, we see in the GeoGebra project a kind of synergy or concerted effort between technology and theory, individual inventions and collective participation, local experiments and global applications. GeoGebra has created a positive ripple effect, centered around technology integration in mathematics teaching and learning, which has reached out from a graduate design project at the University of Salzburg across international borders to all major regions of the world, from university students to children in rural areas. For the most part, GeoGebra and GeoGebra-based curricular activities have been a grassroots phenomenon, motivated distinctively by teachers' professional commitment and their mathematical and didactical curiosity.

This volume stands as an initial endeavor to survey GeoGebra-inspired educational efforts or experiments in both theory and practice in mathematics education across the grade levels. The focus of the book is centered on the international use of GeoGebra in model-centered mathematics teaching and learning, which naturally goes beyond traditional mathematics instruction in content and coverage of concepts. The chapters in this volume address broad questions of mathematics education, citing specific examples along the way, with a clear commitment to mathematical understanding and mathematical applications. In addition to being a computational tool, GeoGebra has been characterized by several authors to be a conceptual tool, a pedagogical tool, a cognitive tool, or a transformative tool in mathematics teaching and learning. This *tool* perspective underlines the versatile roles of GeoGebra in mathematical instruction and mathematics education reforms. In general, the chapters address mathematics teaching and learning as a complex process, which calls for technological tools such as GeoGebra for complexity management, multiple representations, sense-making, and decision-making. In what follows, we briefly introduce the key ideas of each chapter along six themes that run naturally through all the chapters.

History, Philosophy, and Theory

In Chapter 1, Hohenwarter and Lavicza review the history and philosophy behind the initial GeoGebra project and its subsequent and ongoing evolution into an international community project. They further envision a community-based approach to technology integration in mathematics education on an international scale. Chapter 2 features a theoretical paper by Bu, Spector, and Haciomeroglu, who review the literature on mathematical understanding from the psychological, philosophical, and mathematical perspectives, shedding light on the relevancy of mental models in reconceptualizing mathematical meaning and understanding. They put forward a preliminary framework for GeoGebra-integrated instructional design by synthesizing major principles from Model-Facilitated Learning, Realistic Mathematics Education, and Instrumental Genesis. The overarching goal is to identify design principles that foster deep mathematical understanding by means of

GeoGebra-based conceptual models and modeling activities. They also call for increased attention to the mutually defining role of GeoGebra tools and students' instrumented mathematical behavior, especially in complexity management.

Dynamic Modeling and Simulations

In Chapter 3, Pierce and Stacey report on the use of dynamic geometry to support students' investigation of real-world problems in the middle and secondary grades. Dynamic models of real-world scenarios, as they found, help students to make mathematical conjectures and enhance their understanding of the mathematical concepts. Furthermore, the multiple features of dynamic modeling contribute to improving students' general attitudes toward mathematics learning.

Burke and Kennedy (Chapter 4) explore the use of dynamic GeoGebra models and simulations in building a bridge between students' empirical investigations and mathematical formalizations. Their approach to abstract mathematics illustrates the didactical conception of vertical mathematization, a process by which mathematical ideas are reconnected, refined, and validated to higher order formal mathematical structures (e.g., Gravemeijer & van Galen, 2003; Treffers, 1987). They aim to provide model-based conceptual interventions that support students' development of valid mental models for formal mathematics, an important practice that typically receives inadequate treatment in upper-division mathematics courses. In Chapter 5, Novak, Fahlberg-Stojanovska, and Renzo present a holistic learning model for learning mathematics by doing mathematics—building simulators with GeoGebra to seek deep conceptual understanding of a real-world scenario and the underlying mathematics (cf. Alessi, 2000). They illustrate their learning model with a few appealing design examples in a setting that could be called a mathematical lab, where science and mathematics mutually define and support one another in sense-making and mathematical modeling.

GeoGebra Use, Problem Solving, and Attitude Change

Iranzo and Fortuny (Chapter 6) showcase, from the perspective of instrumental genesis, the complex interactions among the mathematical task, GeoGebra tool use, and students' prior mathematical and cognitive background, citing informative cases from their study. GeoGebra-based modeling helped their students diagnose their mathematical conceptions, visualize the problem situations, and overcome algebraic barriers and thus focus on the geometric reasoning behind the learning tasks. Students' problem solving strategies, as the authors observe, are the result of the nature of the instructional tasks, students' background and preferences, and the role of the teacher. In Chapter 7, Mousoulides continues the discussion about the modeling approach to GeoGebra-integrated problem solving in the middle grades, where GeoGebra is employed as a conceptual tool to help students make connections between real-world situations and mathematical ideas. Students in his study constructed sophisticated dynamic models, which broadened their mathematical exploration and visualization skills.

Chapter 8 features an article by Arranz, Losada, Mora, Recio, and Sada who report on their experience in modeling a 3-D linkage cube using GeoGebra. In the process of building a GeoGebra-based flexible cube, one encounters interesting connections between geometry and algebra and develops problem solving skills while resolving intermediate challenges along the way. The cube problem and its educational implications are typical of a wide range of real-world modeling problems in terms of the mathematical connections and the ever expanding learning opportunities that arise, sometimes unexpectedly, in the modeling process (e.g., Bu, 2010).

Haciomeroglu (Chapter 9) reports on his research on secondary prospective teachers' experience with GeoGebra-based dynamic visualizations in instructional lesson planning. His findings highlight the impact of GeoGebra use on participants' attitudes toward mathematic teaching and the importance of collaborative group work in GeoGebra-integrated teacher education courses.

Gómez-Chacón (Chapter 10) adopts a multi-tier, mixed methods research design, which consists of a large-scale survey ($N = 392$), a small focus study group ($N = 17$), and six individual students, to investigate the influences of GeoGebra-integrated mathematics instruction on secondary students' attitudes toward mathematics learning in computer-enhanced environments. While GeoGebra use is found to foster students' perseverance, curiosity, inductive attitudes, and inclination to seek accuracy and rigor in geometric learning tasks, the findings also point to the complex interactions between computer technology, mathematics, and the classroom environment. The author further analyzes the cognitive and emotional pathways underlying students' attitudes and mathematical behaviors in such instructional contexts, calling for further research to find ways to capitalize on the initial positive influences brought about by GeoGebra use and foster the development of students' sustainable positive mathematical attitudes.

GeoGebra as Cognitive and Didactical Tools

Karadag and McDougall (Chapter 11) survey the features of GeoGebra from the cognitive perspective and discuss their pedagogical implications in an effort to initiate both theoretical and practical experimentation in conceptualizing GeoGebra as a cognitive tool for facilitating students' internal and external multiple representations (cf. Jonassen, 2003; Jonassen & Reeves, 1996). Along a similar line of thought, Ronchi (Chapter 12) views GeoGebra as a methodological or didactical resource that supports the teaching and learning of mathematics by helping teachers and their students visualize formal mathematical knowledge and promote their sense of ownership through dynamic constructions in a lab setting.

Curricular Initiatives

In Chapter 13, Little outlines his vision for a GeoGebra-based calculus program at the high school level, showcasing the distinctive features of GeoGebra for facilitating students' and teachers' coordination of algebra and geometry, which is at the very core of learning and teaching calculus. As seen by Little, the simplicity of GeoGebra's user interface and its computational architecture allow students to construct their own mathematical models and, by doing so, reinvent and enhance their ownership of calculus concepts. In Chapter 14, Lingefjärd explores the prospect of revitalizing Euclidean geometry in school mathematics in Sweden and internationally by taking advantage of GeoGebra resources. Perhaps, a variety of school mathematics, including informal geometry and algebra, can be reconsidered and resequenced along Little and Lingefjärd's lines of thought. In response to increased computational resources and the evolving needs of society (exemplified often by applications of number theory, for example), our conception of mathematics has changed significantly over the past several decades. It is likely that the open accessibility and the dynamic nature of GeoGebra may contribute to or initiate a similarly profound evolution of school mathematics and its classroom practice.

Equity and Sustainability

GeoGebra has also inspired research and implementation endeavors in developing countries, where access to advanced computational resources is limited. In Chapter 15, De las Peñas and Bautista bring the reader to the Philippines to observe how children and their mathematics teachers coordinate the construction of physical manipulatives and GeoGebra-based mathematical modeling activities. They also share their approaches to strategic technology deployment when a teacher is faced with limited Internet access or numbers of computers. Jarvis, Hohenwarter, and Lavicza (Chapter 16) reflect on the feedback from international users of GeoGebra and highlight a few key characteristics of the GeoGebra endeavor—its dynamic international community, its sustainability, and its values in providing equitable and democratic access to powerful modeling tools and mathematics curricula to all students and educators across the world. As GeoGebra users join together with mathematicians and mathematics educators, the authors call for further research on the development of GeoGebra-inspired technology integration and the influence and impact of GeoGebra and the GeoGebra community in the field of mathematics education.

It is worth noting that, given the international nature of this first volume on GeoGebra and its applications in mathematical modeling, the editors encountered great challenges in the editing process in terms of languages and styles. With certain manuscripts, extensive editorial changes were made by the editors and further approved by the chapter authors. Meanwhile, the editors tried to maintain the international flavor of the presentations. We invite our readers to consider the context of these contributions, focus on the big ideas of theory and practice, and further join us in the ongoing experimentation of community-based technology integration in mathematics education, taking advantage of GeoGebra and similar technologies.

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Lingguo Bu
Department of Curriculum and Instruction
Southern Illinois University Carbondale, IL, USA

Robert Schoen
Florida Center for Research in Science, Technology, Engineering, & Mathematics
Florida State University, Tallahassee, FL, USA

MARKUS HOHENWARTER AND ZSOLT LAVICZA

1. THE STRENGTH OF THE COMMUNITY: HOW GEOGEBRA CAN INSPIRE TECHNOLOGY INTEGRATION IN MATHEMATICS

The dynamic mathematics software GeoGebra has grown from a student project into a worldwide community effort. In this chapter, we provide a brief overview of the current state of the GeoGebra software and its development plans for the future. Furthermore, we discuss some aspects of the fast growing international network of GeoGebra Institutes, which seeks to support events and efforts related to open educational materials, teacher education and professional development, as well as research projects concerning the use of dynamic mathematics technology in classrooms all around the world.

INTRODUCTION

During the past decades, it has been demonstrated that a large number of enthusiasts can alter conventional thinking and models of development and innovation. The success of open source projects such as Linux®, Firefox®, Moodle®, and Wikipedia® shows that collaboration and sharing can produce valuable resources in a variety of areas of life. With the increased accessibility of affordable computing technologies in the 1980s and 90s, there was overly enthusiastic sentiment that computers would become rapidly integrated into education, in particular, into mathematics teaching and learning (Kaput, 1992). However, numerous studies showed only a marginal uptake of technology in classrooms after more than two decades (Gonzales, 2004). There were many attempts and projects to promote wider technology integration, but many of these attempts led to only marginal changes in classroom teaching (Cuban, Kirkpatrick & Peck, 2001). While working on the open source project GeoGebra, we are witnessing the emergence of an enthusiastic international community around the software. It will be interesting to see whether or not this community approach could penetrate the difficulties and barriers that hold back technology use in mathematics teaching. Although the community around GeoGebra is growing astonishingly fast, we realize that both members of the community and teachers who are considering the use of GeoGebra in their classrooms need extensive support. To be able to offer such assistance and promote reflective practice, we established the International GeoGebra Institute (IGI) in 2008. In this chapter, we offer a brief outline of the current state of both the GeoGebra software and its community, and we also hope to encourage colleagues to join and contribute to this growing community.

L. Bu and R. Schoen (eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra, 7–12.

GEOGEBRA

The software GeoGebra originated in the Master's thesis project of Markus Hohenwarter at the University of Salzburg in 2002. It was designed to combine features of dynamic geometry software (e.g., Cabri Geometry®, Geometer's Sketchpad®) and computer algebra systems (e.g., Derive®, Maple®) in a single, integrated, and easy-to-use system for teaching and learning mathematics (Hohenwarter & Preiner, 2007). During the past years, GeoGebra has developed into an open-source project with a group of 20 developers and over 100 translators across the world. The latest version of GeoGebra offers dynamically linked multiple representations for mathematical objects (Hohenwarter & Jones 2007) through its graphical, algebraic, and spreadsheet views. Under the hood, we are already using a computer algebra system (CAS) that will be made fully accessible for users through a new CAS view in the near future. GeoGebra, which is currently available in 50 languages, has received several educational software awards in Europe and the USA (e.g. EASA 2002, digita 2004, Comenius 2004, eTwinning 2006, AECT 2008, BETT 2009 finalist, Tech Award 2009, NTLC Award 2010).

Apart from the standalone application, GeoGebra also allows the creation of interactive web pages with embedded applets. These targeted learning and demonstration environments are freely shared by mathematics educators on collaborative online platforms like the GeoGebraWiki (www.geogebra.org/wiki). The number of visitors to the GeoGebra website has increased from about 50,000 during 2004 to more than 5 million during 2010 (see Figure 1) coming from over 180 countries.

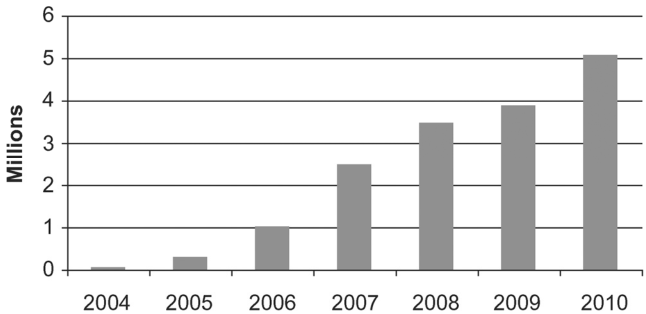

Figure 1. Visitors per year to www.geogebra.org (in millions).

INTERNATIONAL GEOGEBRA INSTITUTE (IGI)

The growing presence of open-source tools in mathematics classrooms on an international scale is calling for in-depth research on the instructional design of GeoGebra-based curricular modules and the corresponding impact of its dynamic mathematics resources on teaching and learning (Hohenwarter & Lavicza, 2007). Thus, we gathered active members of the GeoGebra community from various countries at a conference in Cambridge, UK in May 2008, and founded an international research and professional development network: the International GeoGebra Institute (www.geogebra.org/igi). This not-for-profit organization intends to coordinate international research and professional development efforts around the free software. The main goals of the International GeoGebra Institute are to:

– Establish self-sustaining local GeoGebra user groups;
– Develop and share open educational materials;
– Organize and offer workshops for educators;
– Improve and extend the features of the software GeoGebra;
– Design and implement research projects both on GeoGebra and IGI;
– Deliver presentations at national and international conferences.

FUTURE AND VISION

In order to provide adequate support and training, we are in the process of establishing local groups of teachers, mathematicians, and mathematics educators who work together in developing and adapting the software as well as educational and professional development materials to serve their local needs. For example, through a recent project funded by the National Centre for Excellence in Mathematics Teaching (NCETM), we have been collaborating with nine mathematics teachers in England to embed GeoGebra-based activities into the English curriculum and develop adequate professional development programs (Jones et al., 2009). This project aspired to nurture communities of teachers and researchers in England who are interested in developing and using open source technology in schools and in teacher education.

Since May 2008, more than forty local GeoGebra Institutes have already been established at universities in Africa, Asia, Australia, Europe, North and South America (Figure 2). For example, the Norwegian GeoGebra Institute in Trondheim comprises of more than 50 people in a nation-wide network of GeoGebra trainers, mathematicians, and mathematics educators who provide support for teachers and collaborate on research projects in relation to the use of free educational resources. Since the first international GeoGebra conference in July 2009 in Linz, Austria, more than a dozen local conferences have been held or scheduled in America, Asia, and Europe. These conferences as well as workshops and local meetings are shared and publicized through a public events calendar on GeoGebra's website (Figure 3). For example, several European countries are collaborating in a recently awarded grant to establish a Nordic GeoGebra Network focusing on joint seminars and conferences.

Several local GeoGebra Institutes are also involved in pioneering projects featuring the use of netbook and laptop computers. For example, three million laptops with GeoGebra preinstalled have just been given out to students by the government of Argentina. The GeoGebra Institute in Buenos Aires is actively involved in corresponding teacher training and curricular development activities.

Similar laptop projects are in progress in Australia and Spain. More information on the different GeoGebra Institutes and their activities can be found on http://www.geogebra.org/igi.

Figure 2. Network of local GeoGebra Institutes: www.geogebra.org/community.

Figure 3. GeoGebra events map and calendar: http://www.geogebra.org/events.

DEVELOPMENT OF INSTRUCTIONAL MATERIALS

On the GeoGebraWiki (www.geogebra.org/wiki) website, users have already shared over fifteen thousand free interactive online worksheets that can be remixed and adapted to specific local standards or individual needs. In order to better support the sharing of open educational materials in the future, we are working on a material sharing platform that will also allow users to provide comments and rate the quality of materials. Furthermore, GeoGebra materials will also be useable on mobile devices and phones in the future (e.g., iPhone®, iPad®, Android® phones, Windows® phones).

Concerning the software development of GeoGebra, we are engaging more and more talented Java programmers with creative ideas for new features and extensions through our new developer site (www.geogebra.org/trac). With the recent addition of a spreadsheet view, GeoGebra is ready for more statistical charts, commands, and tools. The forthcoming computer algebra system (CAS) and 3D graphics views will provide even more applications of the software both in schools as well as at the university level. With all these planned new features, it will be crucial to keep the software's user interface simple and easy-to-use. Thus, we are also working on a highly customizable new interface where users can easily change perspectives (e.g., from geometry to statistics) and/or rearrange different parts of the screen using drag and drop.

OUTREACH

As an open source project, GeoGebra is committed to reaching out specifically to users in developing countries who otherwise may not be able to afford to pay for software. Together with colleagues in Costa Rica, Egypt, the Philippines, Uruguay, and South Africa, we are currently investigating the possibilities of setting up local user groups or GeoGebra Institutes, and developing strategies to best support local projects in these regions. For example, we have recently developed a special GeoGebra version for the one-laptop-per-child project in Uruguay. Involving colleagues in our international network could create new opportunities to support countries with limited resources and exchange educational resources and experiences.

SUMMARY

With this introductory chapter, we hope to raise attention to the growing GeoGebra community and encourage our colleagues in all nations to contribute to our global efforts in enhancing mathematics education for students at all levels. It is fascinating and encouraging to read about the various approaches our colleagues have taken to contribute to the GeoGebra project. If you are interested in getting involved in this open source endeavor, please visit the GeoGebra/IGI websites, where we will continue to discuss together which directions the GeoGebra community should take in the future.

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Markus Hohenwarter
Mathematics Education
Johannes Kepler University, Austria

Zsolt Lavicza
Faculty of Education
University of Cambridge, United Kingdom

LINGGUO BU, J. MICHAEL SPECTOR, AND
ERHAN SELCUK HACIOMEROGLU

2. TOWARD MODEL-CENTERED MATHEMATICS LEARNING AND INSTRUCTION USING GEOGEBRA

A Theoretical Framework for Learning Mathematics with Understanding

This chapter presents a model-centered theoretical framework for integrating GeoGebra in mathematics teaching and learning to enhance mathematical understanding. In spite of its prominence in the ongoing mathematics education reform, understanding has been an ill-defined construct in the literature. After reviewing multiple perspectives from learning theories and mathematics education, we propose an operational definition of understanding a mathematical idea as having a dynamic mental model that can be used by an individual to mentally simulate the structural relations of the mathematical idea in multiple representations for making inferences and predictions. We further recognize the complexity of mathematical ideas, calling for a model-centered framework for instructional design in dynamic mathematics. Synthesizing theoretical principles of Realistic Mathematics Education, Model-Facilitated Learning, and Instrumental Genesis, we contend that GeoGebra provides a long-awaited technological environment for mathematics educators to reconsider the teaching and learning of school mathematics in terms of the human nature of mathematics, contemporary instructional design theories, and the influences of digital tools in mathematical cognition. We present three design examples to illustrate the relevance of a model-centered theoretical framework.

INTRODUCTION

Mathematics learning and instruction is a highly complex process as has been unveiled by more than three decades of research in mathematics education (Gutiérrez & Boero, 2006; Lesh, 2006; Lesh & Doerr, 2003). Under the surface of symbols and rules lies a rich world of mathematical ideas that permeate a host of contexts and various domains of mathematics. The cognitive complexity of mathematics in general reflects the human nature of mathematics and mathematics learning and instruction that can be characterized in multiple dimensions (Dossey, 1992; Freudenthal, 1973). First, mathematics learning is both an individual and a social process, where diverse ways of individual experiences interact with the normative elements of a field with thousands of years of history. Second, there are virtually no isolated mathematical ideas. From numeration to calculus, each mathematical concept is connected to other

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concepts and vice versa. `It is within such a web of connected concepts that each mathematical idea takes on its initial meaning and further evolves as learners come into closer contact with a variety of related concepts and relations. Third, these interconnections among mathematical ideas are frequently solidified by their multiple representations and the connections among the multiple representations (Goldin, 2003; Sfard, 1991). A parabola, for example, is connected to and further understood in depth by virtue of its relations to lines, points, conics, squares, area, free fall, paper-folding, projectiles, and the like. It is further represented by verbal, numeric, algebraic, and geometric representations, and in particular, their interconnections. Fourth, mathematical representations are ultimately cultural artifacts, indicative of the semiotic, cultural, and technological developments of a certain society (Kaput, 1992; Kaput, Hegedus, & Lesh, 2007; Presmeg, 2002, 2006). For example, although the abacus has been used in some Asian cultures for centuries as a primary calculation device, it now coexists with graphing calculators and computer software. Technology changes, and it further changes what we do and what we can do as well as the way we handle traditional instructional practices (Milrad, Spector, & Davidsen, 2003). With a growing variety of new tools available for mathematics learning and teaching, traditionally valued mathematical operations such as graphing and factoring are becoming trivial mathematical exercises; learners and teachers alike are faced with new choices with regard to the use of tools and the redesign of learning activities (Puntambekar & Hubscher, 2005). All these aspects of mathematics education contribute to its growing complexity, only to be further complicated by the evolving role of mathematics and changing goals of mathematics education in an ever-changing information society (diSessa, 2007; Kaput, Noss, & Hoyles, 2002).

The complexity of mathematics learning and instruction lends itself to a variety of theoretical frameworks and new interactive learning technologies. The theory of Realistic Mathematics Education (RME) (Freudenthal, 1978; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Streefland, 1991; Treffers, 1987) stands out among the contemporary theories of mathematics education because it is grounded in the historical and realistic connections of mathematical ideas. RME conceptualizes mathematics learning as a human activity and a process of guided reinvention through horizontal and vertical mathematizations. In horizontal mathematization, realistic problem situations are represented by mathematical models in a way that retains its essential structural relations; in vertical mathematization, these models are further utilized as entry points to support sense-making within a world of increasingly abstract mathematical ideas in a chain of models. Within RME, models are used primarily as didactical tools for teaching mathematics to situate the origin and the conceptual structure of a mathematical idea (Van den Heuvel-Panhuizen, 2003). However, with natural extensions, such didactical models can be used to generate more advanced ideas and foster problem solving skills, especially in vertical mathematization. The instructional principles of RME are further supported by new interactive mathematics learning technologies, which typically provide multiple representations, dynamic links, and simulation tools. Among the various mathematics learning technologies, GeoGebra (www.geogebra.org) has gained growing international recognition since its official release in 2006 because of its open source status, international developers, and a growing user base of mathematicians, mathematics educators, and classroom teachers (J. Hohenwarter & M. Hohenwarter, 2009; Hohenwarter & Preiner, 2007). As a 21st-century invention, GeoGebra is one of several next-generation mathematics learning technologies that are reshaping the representational infrastructure of mathematics education and providing the world community with easy and free access to powerful mathematical processes and tools (Kaput et al., 2002).

Viewed from the theoretical perspective of RME, GeoGebra affords a variety of digital resources that allow learners to mathematize realistic problem situations, invent and experiment with personally meaningful models using multiple representations and modeling tools, and further proceed to formulate increasingly abstract mathematical ideas. GeoGebra is open source and thus is freely available to the international community; it is also Web-friendly and is thus supportive of both individual reflection and Web-based social interactions. This integration of RME principles and GeoGebra technological features finds a similar theoretical framework developed in the instructional design community—Model-Facilitated Learning (MFL) (de Jong & van Joolingen, 2008; Milrad et al., 2003). As a technology-integrated instructional design framework grounded in Model-Centered Learning and Instruction (MCLI) (Seel, 2003, 2004), MFL tackles complex subject matter through modeling and simulations using systems dynamic methods and emphasizing the use of concrete scenarios, complexity management, and high-order decision-making. The existence of GeoGebra provides an intellectual bridge that connects a domain-specific theory of mathematics education, RME, and a general instructional design framework that is grounded in contemporary learning theories. Indeed, Seel (2003) characterizes RME as one of the exemplary domain-specific theories that operationalizes the basic tenets of MCLI. In our efforts to seek a theoretical framework that facilitates GeoGebra-integrated mathematics learning and instruction, we found it useful to synthesize RME and MFL principles, incorporating recent developments in the use of technology in mathematics education, in particular, the theory of Instrumental Genesis (IG) (Guin, Ruthven, & Trouche, 2005; Trouche, 2004), which sheds light on the mutually defining relationship between technology use and learners' evolving ways of mathematical reasoning. We believe that these three theoretical frameworks, in spite of their different origins and theoretical orientations, are collectively informative with regard to the ongoing use of GeoGebra in mathematics education.

In this chapter, we synthesize the major principles of RME and MFL in an effort to develop a preliminary theoretical framework toward model-centered learning and instruction using GeoGebra. We recognize both the didactical and the mathematical complexity of subject matter and the integral role of technology in mathematics learning and teaching, aiming for deep mathematical understanding and meaningful learning.

UNDERSTANDING AND DYNAMIC REPRESENTATIONS

A recurring and dominant theme in mathematics education reform is *understanding*, which is frequently used in conjunction with sense-making or meaningful learning in such phrases as *teaching for understanding* and *learning with understanding* (Brenner et al., 1997; Darling-Hammond et al., 2008). Understanding has, in effect, become a means and a goal of mathematics education. However, there is no clear definition of mathematical understanding. By contrast, it is relatively easy to identify specific cases where learners show a lack of understanding. For example, some students may automatically resort to subtraction in response to □ + 7 = 21, but cannot explain why they did that or if their answer 14 is correct. Similar examples are abundant in school mathematics.

Johnson-Laird (1983) suggests that the term *understanding* has plenty of criteria but may not have an essence. In his theory of comprehension, he contends that in understanding an utterance, learners first construct propositional representations and further make use of such propositional representations for the construction of a mental model, which preserves the structural relations in a state of affairs and enables the learner to make inferences. Mental models can be recursively revised and dynamically manipulated in support of deeper comprehension and inferences. Our understanding of a certain phenomenon amounts to the construction of a mental model of it; our interpretation depends on *both* the model *and* the processes involved in the construction, extension, and evaluation of the mental model. Indeed, as Johnson-Laird (1983) argues, "all our knowledge of the world depends on our ability to construct models of it" (p. 402).

Johnson (1987) examines understanding from the perspective of embodied cognition and describes understanding as "an event in which one has a world, or, more properly, a series of ongoing related meaning events in which one's world stands forth" (p. 175). He characterizes meaning as a matter of understanding, which is always about relatedness as a form of intentionality in that "[a]n event becomes meaningful by pointing beyond itself to prior event structures in experience or toward possible future structures" (p. 177). Johnson conceives image schemata as organizing mental structures for human experience and understanding. Image schemata, which are functionally similar to Johnson-Laird's (1983) mental models, are dynamic in nature because they are conceived to be flexible structures of activities. In summary, according to Johnson (1987), understanding is "an evolving process or activity in which image schemata, as organizing structures, partially order and form our experience and are modified by their embodiment in concrete experiences" (p. 30).

Furthermore, Perkins (1986) approaches understanding from his theory of *knowledge as design*, calling our attention to the metaphorical meaning of the term understanding. To understand means *to stand under* or be an insider of a problem situation. Our knowledge of a situation is accordingly a matter of design that has a purpose, a structure, model cases, and related arguments. Conceptual models, in particular, mediate human understanding, where mental models play a critical role, pervading, enabling or even disabling the cognitive processes.

The brief review above points to a set of common criteria for understanding as conceived in the fields of psychology, philosophy, and learning sciences:

– Understanding of a situation relies on a mental model that preserves the relevant or salient structural relations of a perceived or intuited state of affairs.
– A mental model is dynamic in nature and evolves with experience.
– The human ability to use mental models involves a system of relations that manages complexity and simulates a situation, enabling us to experience meaning and make inferences (cf. Seel, 2003).

Mental models are internal structures that are formulated in one's mind. But where do they come from? Johnson-Laird (1983) suggests that mental models are originally constructed through one's perceptual experience of the world, depending "both on the way the world is and on the way we are" (p. 402). A mental model therefore plays the role of a mental world that connects human imagination and the outside world. Johnson's (1987) image schemata are conceived as "recurring structures of, or in, our perceptual interactions, bodily experience, and cognitive operations" (p. 79). Perkins (1986) also recognizes the central role of mental models in framing our understanding, arguing that both mental and physical models are designs that are necessary components of human knowledge acquisition. Along the same line of thought, Norman (1983) regards mental models as naturally evolving models of a target system, which are not necessarily accurate but are functional in enabling people to make decisions or predications. As such, people's mental models also include their beliefs about themselves as well as the target system.

It follows, accordingly, that our understanding of a mathematical topic is a matter of having a functional mental model for it. Such a mental model does not only represent internally the state of relations of the mathematical topic but also runs dynamically in support of problem solving, including making wrong inferences (Norman, 1983; Seel, Al-Diban, & Blumschein, 2000). As internal entities, mental models cannot be directly assessed or constructed in an instructional setting. To assess one's mental models, it is necessary to have them externalized by means of cultural artifacts such as linguistic resources and mathematical notations. To support learners' construction of mathematically viable mental models, instructional designers need to provide model-eliciting activities, including intellectually appropriate conceptual models. In either direction, this leads to the discussion of multiple representations and conceptual and procedural understanding in mathematics teaching and learning.

Mathematics is a system of ideas developed over centuries as an outcome of the individual and collective endeavor of human experience (Dossey, 1992). The abstract nature of a mathematical idea is much similar to that of a mental model, which is not surprising at all, since one's mathematical ideas are mental models. Just as a mental model has two major components (i.e., a structure that preserves the relevant relations and the corresponding processes that allow the model to run dynamically), a mathematical idea is conceived as an interplay between one's conceptual and procedural knowledge (Hiebert & Carpenter, 1992; Silver, 1986). For historical reasons, mathematics has been taught with too much emphasis on its procedural aspects, resulting in a host of learning problems among students who can perform some procedures correctly, but are little aware of what they have done and why their result may be correct (National Council of Teachers of Mathematics [NCTM], 2000). Reform efforts since the 1980s have explicitly called for the pedagogical coordination of the two aspects of mathematical knowledge, especially in problem solving situations (Silver, 1986). To understand a mathematical idea therefore is to have a mental model that integrates both its conceptual and its procedural aspects.

In light of the complexity of mathematical ideas and the limitations and affordances of mathematical representations, mathematical understanding has accordingly been characterized as a person's ability to navigate through a system of multiple representations such as verbal expressions, diagrams, numeric tables, graphics, and algebraic notations (Goldin, 2003; Hiebert & Carpenter, 1992) and to grasp the relationships among the various representations and their structural similarities and differences (Goldin & Shteingold, 2001). This emphasis on multiple representations in mathematics education is consistent with similar principles involving complex subject matter in the learning sciences (Milrad et al., 2003; Minsky, 2006) since each representation carries its own limitations as well as affordances. From a practical perspective, if a learner can coordinate a variety of representations as a mathematically competent person does, there is solid evidence that he or she understands or, in other words, has a valid mental model for the underlying mathematical idea. Furthermore, each representation, such as a table or a graph, is characterized as the totality of a product and the related processes, which refers to "the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM, 2000, p. 67), including both external and internal representations. Thus, each representation ought to be conceived as a mental model on the part of the learner, which is used to recursively transform his or her mental model for the mathematical concept. For example, to understand a linear relation, a learner should be encouraged to seek a comprehensive mental model that synthesizes the underlying structure behind its verbal descriptions, problem situations, numeric tables, graphs, algebraic expressions, and the various connections among them. A graph, as a constituent sub-model, represents the linear nature of the relationship. It also facilitates the corresponding procedures such as graphing, calculating its slope, finding inverses, and making predictions. When a learner's mental model for the graph is enriched through experience, the comprehensive mental model is recursively enhanced. Understanding thus occurs as the learner constructs increasingly mature mental models of the mathematical idea. There is not a clear endpoint in most cases.

When multiple representations are utilized to illustrate various aspects of a mathematical idea, they contribute to the complexity of the learning environment and the cognitive load on the part of learners. Learners who seek deep understanding are expected to grasp not only the dynamic nature of each representation but also the dynamic connections among the multiple representations. Attaining such a level of mathematical understanding, which exists in the mind of mathematically proficient learners, is a daunting endeavor in traditional educational settings since it typically spans a long period of time. The invention of dynamic and interactive technologies, however, has reshaped the representational infrastructure of mathematics, allowing for personally identifiable dynamic representations and, more importantly, automated linking of multiple representations (Hegedus & Moreno-Armella, 2009; Kaput, 1992; Moreno-Armella, Hegedus, & Kaput, 2008). The interactive nature of new technologies further support and constrain the co-actions between learners and the target system (Moreno-Armella & Hegedus, 2009), establishing a kind of partnership of cognition (Salomon, Perkins, & Globerson, 1991). The interplay between dynamic representations and mathematical ideas further enhances the social communication about mathematics, leading to discoveries of pedagogically powerful "synergies between representations and concepts" and "conceptually better-adapted versions of old ones" (diSessa, 2007, p. 250).

In summary, our understanding of a mathematical idea depends on a viable mental model that captures its structural relations and the corresponding processes. Given the complexity of mathematics, it is essential that learners interact with and construct its multiple representations. These multiple representations can be separately constructed and manipulated and also dynamically coordinated using emergent learning technologies, such as GeoGebra, in an environment that supports co-actions between the learner and mathematical representations. In theory, dynamic representations are well aligned with our conception of mental models as the foundation of mathematical understanding and are typical of the behavior of mathematically proficient learners (Nickerson, 1985). In practice, however, dynamic multiple representations pose serious challenges to instructional design. Given the complexity of mathematical ideas, mathematics instruction calls for a starting point that gradually guides learners' development of increasingly powerful and complex mathematical understanding. Thus, teaching mathematics using dynamic technology is an instructional design problem. In the next section, we discuss instructional design principles that may support learners' mathematical development when dynamic mathematics learning technologies are integrated as infrastructural representations (diSessa, 2007).

MODEL-FACILITATED LEARNING FOR DYNAMIC MATHEMATICS

Technology is becoming pervasively influential in mathematics education in that it is playing a "fundamental yet invisible role" (Kaput et al., 2007, p. 190) in much the same way that electricity, mobile phones, and emails are pervasive and influential and mostly taken for granted, especially when readily available and in good working order. Our teaching practices and beliefs about teaching and learning traditional mathematics are facing challenges from new technological tools such as WolframAlpha® (www.wolframalpha.com) and open-source environments like GeoGebra (www.geogebra.org). Indeed, virtually all traditional K-12 mathematical problems can now be readily solved by WolframAlpha®, which accepts natural-language input and also provides a host of related concepts and representations. Predictably, the technologies are getting more intuitive and powerful. Indeed, the very goal of mathematics education is challenged by these technologies. If our primary goal for school mathematics were to enable children to solve those problems, then there would not be much they need to know beyond software navigation skills. By contrast, if our goal is to empower children to understand mathematics in the sense of having a valid, culturally acceptable mental model for making decisions, judgments, and predictions, then there is very little such technical tools can offer to school children. Those tools are powerful and informative resources, but the rest belongs to careful instructional design and classroom implementation.

In recognition of the epistemic complexity of mathematics (Kaput et al., 2007) and the generative power of dynamic learning technologies, we contend that Model-Facilitated Learning (MFL) (Milrad et al., 2003) can be adopted as an overarching framework for reconceptualizing mathematics instruction that takes advantage of emergent dynamic technologies. We further seek domain-specific principles from the theory of Realistic Mathematics Education (RME) (Freudenthal, 1973; Streefland, 1991; Treffers, 1987) and the instrument-related perspectives from the theory of Instrumental Genesis (IG) (Guin et al., 2005).

Model-Facilitated Learning

Decades of research and development in instructional design have identified a few fundamental principles of learning and instruction. Noticeably, understanding is grounded in one's experience; meaning is situated in a context; and learning occurs when changes are made in an integrated system of constituents (Spector, 2004). As a theoretically grounded framework, Model-Facilitated Learning (MFL) (Milrad et al., 2003) draws on such basic principles, well-established learning theories, and methods of system dynamics to manage complexity in technology-enhanced learning environments. MFL seeks to promote meaningful learning and deep understanding, or a systems view of a complex problem situation. The MFL framework consists of modeling tools, multiple representations, and system dynamics methods that allow learners to build models and/or experiment with existing models as part of their effort to understand the structure and the dynamics of a problem situation. MFL recommends that learning be situated in a sequence of activities of graduated complexity, progressing from concrete manipulations to abstract representations while learners are engaged in increasingly complex problem solving tasks. Through the use of multiple representational tools, MFL further maintains the transparency of the underlying models that drives the behavior of a system simulation.

As an emergent theoretical framework for instructional design, MFL represents a well-grounded response to the affordances of new technologies and the needs to engage learners in the exploration of complex problems. As advanced instructional technologies are integrated into the teaching and learning of mathematics, and mathematics instruction is integrated with other disciplines of science, mathematics educators and instructional designers find themselves facing similar issues that MFL promises to address. In particular, the MFL component of policy development holds promise in fostering learner reflection and resolving the validation issue raised by Doerr and Pratt (2008), who found that in a virtual modeling activity, learners tended to validate their emergent understanding solely within the virtual domain without connecting it back to the starting problem scenario.

Didactical Phenomenology and Realistic Mathematics Education

Along with other researchers (e.g., Brown & Campione, 1996; Merrill, 2007) in learning and instruction, we contend that effective instructional design starts with a deep understanding of the content knowledge. In mathematics education, Freudenthal's (1983) didactical phenomenology of mathematical structures serves as a theoretical lens through which we can analyze a mathematical concept, including its historical origin, its realistic connections, its extensions, and its learning-specific characteristics. Such analysis lays the foundation for model-based instructional design and further yields a learning trajectory that starts from "those phenomena that beg to be organized and from that starting point teach[es] the learner to manipulate these means of organizing" (Freudenthal, 1983, p. 32).

Along such a learning trajectory, various representations are necessary for the learners to describe and communicate their experiences of the phenomena. This is where the new technologies come into play and facilitate the realization of learning potentials. The new dynamic technologies provide not only traditional forms of representation but also dynamic links and transformations. The dynamic links and transformations are significant in that it captures the dynamic *process* of representation as well as the static *product* of representation.

Freudenthal's didactical phenomenology lays the foundation for the theory of Realistic Mathematics Education (RME), which has at its core the principle that mathematics is a human activity, in which students make sense of realistic problem situations, re-inventing mathematical ideas under the guidance of competent instructors, and gradually creating increasingly abstract mathematical ideas. In such a process of mathematization, students are engaged in the use of a chain of models, which evolves from models *of* concrete learning tasks to models *for* abstract mathematical structures (Gravemeijer et al., 2000).

Instrumental Genesis

Tool use is an essential component of mathematical learning. As learners make use of tools, including both traditional and digital tools, to facilitate their mathematical activities, such tools and their uses also constitute their mental world. A technical tool, which organizes and facilitates an activity, may eventually be internalized as a psychological tool, or rather, an instrument that mediates a learner's mental processes (Vygotsky, 1978). In other words, a technical tool may become a part of a mental model that enables a learner to make inferences in a problem situation. Mariotti (2002) characterizes such an instrument as an internal construction of an external object, which is "the unity between an object … and the organization of possible actions, the utilization schemes that constitute a structured set of invariants, corresponding to classes of possible operations" (p. 703). A common example is the relationship between students' use of the compass as a circle construction tool and their conception of a circle, which automatically has a center and a radius, but makes it difficult for students to grasp other properties of a circle such as those concerning the diameter being the longest chord or locating the missing center of a given circle.

Within the context of a mathematical activity, the interactions between a tool or artifact and the learner are captured in the notion of *Instrumental Genesis* (IG), in which the learner builds his or her own schemes of action for the tool in a process called *instrumentalization* and the tool also shapes the learner's mental conception of the tool and the activity in a process called *instrumentation* (Guin & Trouche, 1999; Hoyles, Noss, & Kent, 2004; Mariotti, 2002; Trouche, 2005). Instrumental genesis is a long-term process that evolves as a learner internalizes more mathematical and technical artifacts and thus becomes more mathematically proficient. In light of the close relationships among the artifacts, the learner, and the specific mathematical activity, it is reasonable to conceptualize instrumental genesis as a triadic theoretical framework that helps make sense of general human activities, including mathematics learning where new tools have become distinctively instrumental (Fey, 2006). In particular, when new dynamic tools are used, the learning outcome is frequently different than the instructor's intentions (Hollebrands, Laborde, & Sträβer, 2008). For example, while the mid-point tool in GeoGebra was intended as an alternative way to find the mid-point of a line segment, we found, in a professional development project in the US Midwest, that some classroom teachers would choose to use it when they were asked to find the mid-point of a segment whose two endpoints were explicitly given as pairs of coordinates.

In summary, instrumental genesis is a kind of descriptive learning theory that has a solid grounding in the social theory of learning and provides a theoretical lens through which we can make sense of learners' use of technological tools and the potential impact of tool use on their mental processes in the context of mathematical activities. It contributes significant ideas to our understanding of mathematics learning in a model-centered perspective, where tools and artifacts are integral components at all stages.

Model-Facilitated Learning (MFL) for Dynamic Mathematics

The notion of dynamic mathematics dates back to Kaput's (1992) conception of representational plasticity when digital media are used to support various forms of representation and has been further developed by other researchers (Moreno-Armella et al., 2008) from historical and epistemological perspectives as new dynamic technological tools become widely accessible. We tend to think of dynamic mathematics as a systematic correlation between the didactical phenomenology of a mathematical idea and the corresponding technological representations and tools. A mathematical idea is dynamic in that it is connected to a variety of other ideas in realistic and mathematical contexts and is also executable in the sense of a mental model and that of a conceptual digital construction. This notion of dynamic mathematics is well aligned with our characterization of mathematical understanding. In a real sense, all mathematical ideas are dynamic in the mind of a mathematically competent person. That process, which takes a long period of time to develop, can be facilitated by new dynamic technologies in support of learners who are on the way to mathematical proficiency. Using Doerr and Pratt's (2008) notation, dynamic mathematics could be conceptualized as a set of (task, tool) pairs, which serve as the modeling infrastructure. Over such a mathematical and technological infrastructure, we seek to apply the MFL principles, thus establishing a preliminary instructional design framework for integrating dynamic technologies into the teaching and learning of mathematics. Our overarching goal is to promote students' deep understanding of mathematics by focusing on the mathematical processes that are involved in problem solving and tool use.

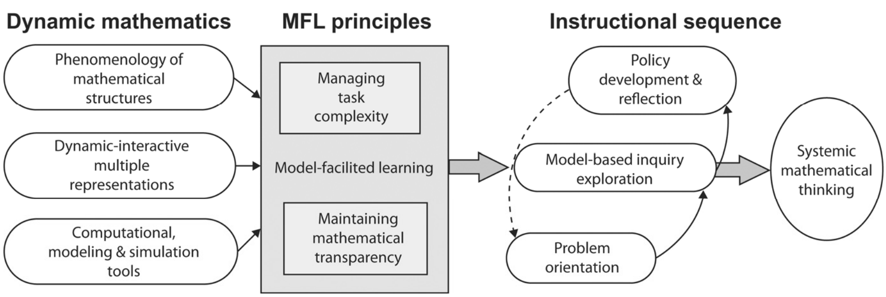

Figure 1. A model-facilitated instructional design framework for dynamic mathematics.

As shown in Figure 1, a lesson design cycle starts with a didactical phenomenological analysis of a mathematical idea, which charts out its structural connections, including its historical, realistic, and formal relations. Next, the technological tools are aligned with the mathematical connections with respect to the global learning objectives. Then, the MFL principles are applied in order to manage the complexity and maintain the mathematical transparency of the mathematical and technological or (task, tool) system. Models of multiple representations and structures play a major role in allowing students to explore and mathematize a starting scenario, develop increasingly abstract understanding, and further make informative decisions about the problem situation. We note that through the stage of policy development or reflection, learners will have an opportunity to examine their modeling activity as a whole, validate or modify their emerging insight into the problem situation. Furthermore, in a modeling environment, the learner development cycle may assume a cyclic form, achieving higher levels of systemic understanding with each cycle. Noticeably, technical tools play an integral role in the modeling process. What initially is a digital representation, a computational utility, or a simulation tool may become integrated into learners' mental world of mathematics as a psychological instrument. Therefore, we should pay special attention to the changing roles of tools in the learning cycle as a way to understand the challenges and opportunities of learners' experience with dynamic mathematics.

Specifically, in app0lying MFL principles to dynamic mathematics, we recommend the following guidelines:

- Conduct a phenomenological analysis of the mathematical idea concerned and identify some historical, realistic, or contemporary problem scenarios to situate the learning process.
- Select a realistic scenario and conduct a thought experiment about the possible stages of learner development and the corresponding scaffolding strategies in what may be called a hypothetical learning trajectory.
- Present problems of increasing complexity and maintain a holistic view of the opening scenario.
- Guide learners in making sense of the problem scenario in mathematical ways such as model building and model use, maintaining awareness of technical tools and their intended functions.
- Challenge learners to examine their modeling process, reflect on the meanings of their tool-enhanced actions, and further develop insight into their actions through decision-making and model-based inquiries.
- Involve learners in group discussions about their learning processes and develop arguments for or against different ways of mathematical thinking and dynamic constructions.

In summary, dynamic mathematics learning technologies, such as GeoGebra, provide an innovative platform to experiment with the basic tenets of Realistic Mathematics Education, in particular, its focus on using realistic contexts as sources of mathematical concepts and guided reinvention as a primary method of mathematization. Guided reinvention involves modeling as a fundamental process of mathematics learning (Gravemeijer & van Galen, 2003). When the complexity of mathematics learning is recognized and further appreciated in the context of emergent digital technological tools, MFL stands as a well-conceived theory-based instructional design framework that addresses the learning of complex subject matter using system dynamics methods and interactive technologies. The theory of instrumental genesis further sheds light on the mutually constitutive relationship between technical tools and learner development. By incorporating the major theoretical principles, we seek to develop a comprehensive design framework to conceptualize the integration of GeoGebra and similar technologies in mathematics teaching and learning. In the next section, we look at three examples that involve the implementation of some of the principles discussed above.

DESIGN EXAMPLES

Quadratic Relations

In this section, we present a model-based learning sequence for the teaching of quadratic relations to algebra students. Quadratic relations exist in various forms in mathematics and are frequently summarized in its algebraic form $f(x) = ax^2 + bx + c$, where a, b, and c are some constants. If $a = 0$, it is reduced to a linear relation. Such a rule, familiar as it is to most algebra students, barely touches on the rich connections and mathematical significance of a quadratic relation. Certainly, we could make use of GeoGebra sliders and animate the effects of a, b, c on the shape and location of the parabola (Figure 2). However, this approach does not add much meaning to the mathematical relation.

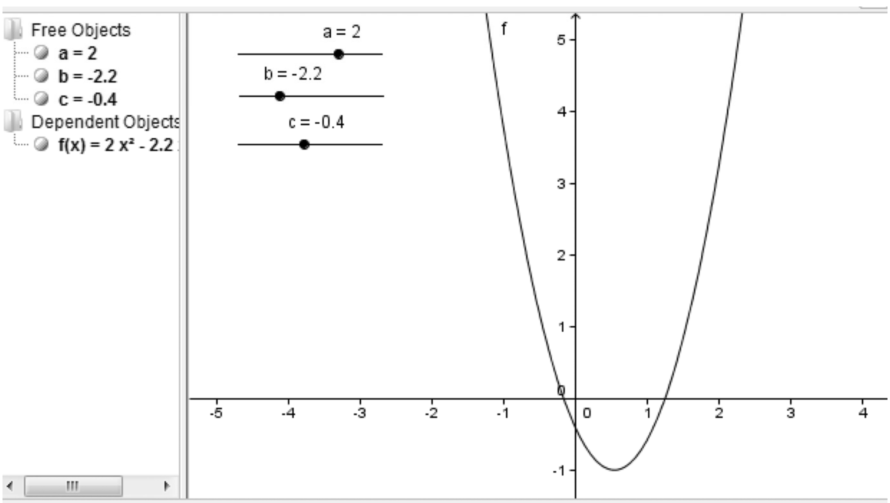

Figure 2. Exploring the effects of a, b, and c on the graph of a quadratic relation.

A preliminary phenomenological analysis of quadratic relations reveals a variety of contexts that may be employed as the foundation for concept formation. First, a quadratic relation can be found in conic sections or similar geometric activities such as paper-folding. Second, it can be found in a context that involves the area of a rectangle with certain width and length. A realistic problem could be stated as: If the width of a rectangle is given as x inches, and the length is two inches longer than the width, how is its area related to the width? Third, a quadratic relation can be found in the real-world construction of a dish antenna, which may use the directrix and focus description of a parabola. Fourth, the quadratic relation can be found in a water fountain or a similar situation involving a projectile. A focal question could be "why does the water stream behave the way it does once it leaves the spout?" Fifth, a quadratic relation exists in the phenomenon of free fall in physics. Other analysis may eventually reveal the fact that a quadratic relation can be developed out of natural or artificial phenomena that involve two dimensions such as width and length for area or time and speed for distance.

Such a phenomenological analysis reveals the complexity of the mathematical idea and justifies the necessity of using multiple models in mathematics instruction. In our work with prospective mathematics teachers, we found the free fall phenomenon interesting since it is naturally familiar to and yet mathematically challenging for most of them. It further provides a context to ground discussion and investigation of multiple related mathematical topics, including constant functions, linear functions, and quadratic functions. Also, the situation could be simplified pedagogically to manage complexity while maintaining the integrity of the whole task. For example, if students find speed change difficult, a sub-problem could be posed for them to model distance changes in the case of an object moving at a fixed speed with no acceleration.

Thus, we can use the free fall phenomenon as a starting point for our discussion of quadratic functions by posing the following problem. The goal is to solve the problem by modeling the scenario and/or derive a formula to solve the problem.

> The Sears/Willis Tower in Chicago is about 442 meters from its roof to the ground. Mark takes a baseball to the roof, and somehow gets it out of the window with no force imposed on it. Now the ball falls freely toward the ground. Assuming that the air has no significant influence on the baseball and the gravitational acceleration in Chicago is approximately 10 m/s^2, Mark wonders, without using calculus: (1) How fast is the ball falling? (2) How does the distance from the roof to the ball change over time?

According to our experience with preservice mathematics teachers, few students had a clear idea of what was going on, although nobody had trouble imagining such a situation. The primary challenge we encountered as instructors was an immediate call for a formula. For various reasons, most students tended to expect a formula to solve a given problem. While a formula does exist in this scenario, it is the least important part of the learning process, at least, under our circumstances. Instead, we could create a dynamic GeoGebra model to make sense of the scenario. The process is rarely sequential, but we need to follow a step-by-step approach in our presentation below.

First, we recognize the fact that there are only three initial parameters involved: the Earth's gravitational acceleration estimated at 10 m/s^2, the height of the tower, and the flow of time. Since we may want to play with these parameters, we choose to use names (or GeoGebra sliders) to represent them. This step is not required, but it leaves room for us to explore the dynamics of the problem. The initial values and the intervals of these sliders can be adjusted according to the real situation.

Second, we want to see how the speed of the baseball changes over time. Some students may choose to graph the function $speed(x) = 10x$, using the x-axis for time. That is a reasonable method if they understand the meaning of the function. However, we choose to graph the speed over time point by point, using the fact that at a given time designated by the slider *Time*, the speed of the ball is *Gravity* × *Time*. Therefore, we can plot the point using a command line input: *Speed = (Time, Gravity * Time)*. This allows students to simulate the situation. When slider *Time* is dragged, the point *Speed* changes accordingly, indicating the change of speed over time. By turning on the *Trace* feature for the point *Speed*, they could see how the speed changes over time (Figure 3). Of course, a point such as *(Gravity, Time)* could also be plotted to visualize the (lack of) change in *Gravity* over time. In the light of the multiple relations in the problem scenario, students should be encouraged, at all stages, to explore their own methods or externalize their conceptions, followed by small-group or whole-class justifications and reflections. For instance, the point-wise graph (Figure 3) can be shown to coincide with the continuous graph of *speed(x) = 10x*, which can be taken advantage of to introduce the meaning of an algebraic function *f(x) = mx*.

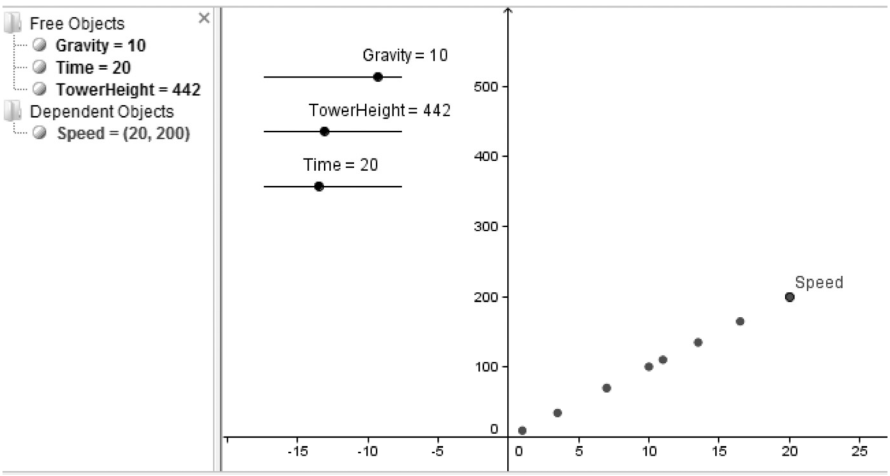

Figure 3. A point-wise plot of the speed-time relation.

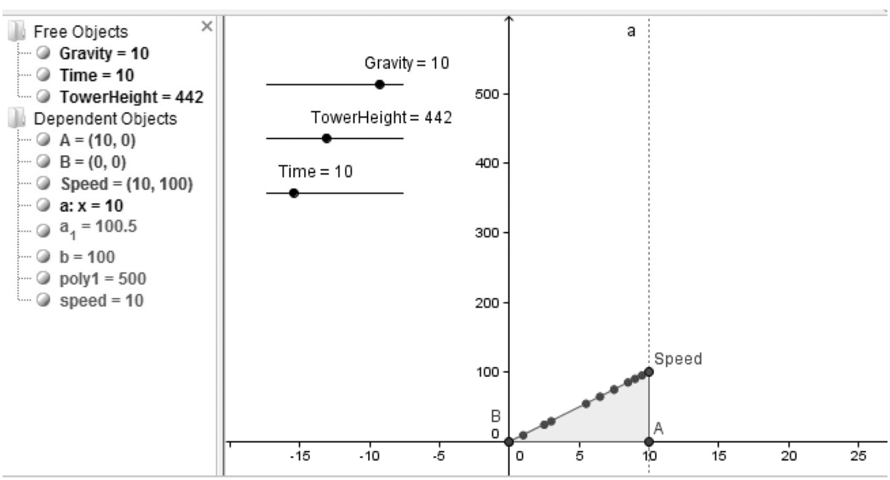

Figure 4. The relationship between distance traveled and area of a triangle.

Third, to find the distance the ball has traveled at a certain time, some cognitive support is necessary, including the use of simpler problems and prompts. For example, the instructor could pose a question about the area of the triangle formed by the points *(0, 0)*, *Speed*, and the corresponding point on the *x*-axis *(x(Speed), 0)*. If necessary, a simpler problem about constant speed and distance could be posed for students to relate distance to area in a geometric way. This critical step represents a cognitive leap and calls for the use of analogical reasoning and, more importantly, social interactions among the students and the instructor, where technology plays a very limited role. Eventually students will come to relate the distance traveled at a certain time to the area of the triangle as shown in Figure 4, and the area tool of GeoGebra thus becomes an instrument for students to find the distance.

Fourth, when students relate the distance traveled to the area of a corresponding triangle, it would be appropriate to ask the question: How is the distance traveled related to time? For that purpose, we can plot a point using the command line input: *Distance = (Time, poly1)*, where *poly1* is the name of the triangle and represents its area. Using the triangle as a whole without calculating its area is one of the features of GeoGebra that supports graduated complexity. At a higher level, it may be very appropriate for students to find an explicit way to calculate the area of the triangle. However, at the current step, the focus is to explore the relationship between distance and time. Using the *Trace* feature for the point *Distance*, students can simulate the free fall process and observe the change of distance over time in addition to the previous speed-time relationship (Figure 5).

Fifth, to find when the ball will hit the ground, we could draw a horizontal line *y = TowerHeight* and simulate the free fall until the *Distance* point goes beyond that line as shown also in Figure 5.

Sixth, since the initial conditions *Gravity* and *TowerHeight* are defined using sliders, students can now change the initial conditions, observe their influences, and ask open-ended questions about the problem scenario. For example, what would happen in a place where gravity is 2.5 m/s^2? What if the gravity is zero? By exploring such questions, students can potentially form a perspective on the problem scenario and identify the structure of the problem (i.e., the constant, linear, and quadratic relations).

Finally, the above GeoGebra sequence could be extended to support higher levels of algebraic thinking. While point-wise graphs represent a snapshot of the underlying structure of the problem scenario, they lack efficiency. As mentioned earlier, some students may find it tempting to graph the speed-time relationship using a function like *speed(x) = Gravity * x*. Along the same line of thinking, they could be guided to find an explicit relation between the distance and the time. In other words, at time *x*, what is the area of the corresponding triangle? Using the base-height rule, at time *x*, the area of the triangle is *(1/2) x * speed(x)*, where *x* is the base and *speed(x)* is the height. Since *speed(x) = Gravity * x*, the distance traveled is (*1/2) Gravity* * x^2. Therefore, we could enter *distance(x) = (1/2) * Gravity * x^2* at the command line (Figure 6). Other dynamic explorations are subsequently possible for students to develop a comprehensive mental model of a quadratic relation and its mathematical connections.

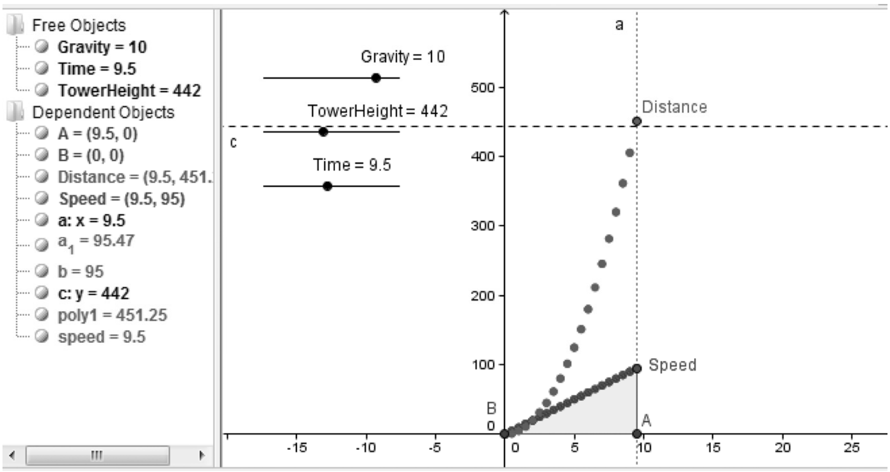

Figure 5. Comparing the relationship between distance and time with the relationship between speed and time.

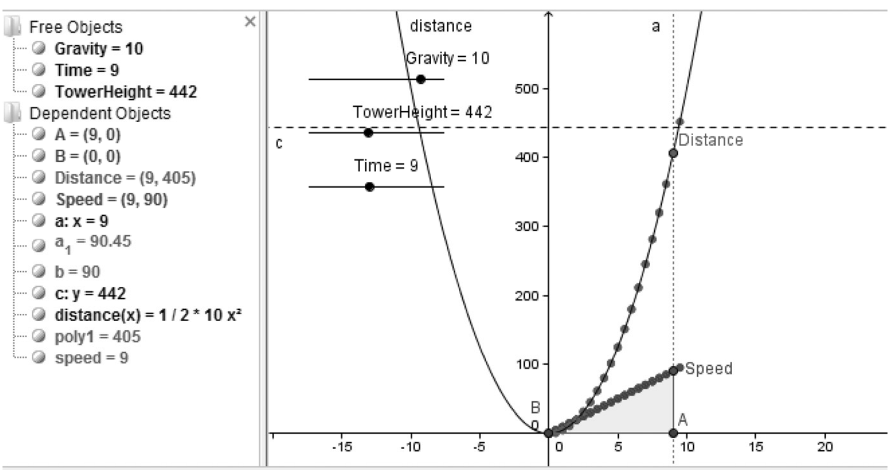

Figure 6. The graph of a quadratic function fits the point-wise simulation of free fall.

To summarize, in the free fall construction we applied the basic principles of RME and MFL in our effort to make sense of not only the problem scenario but, more importantly, the mathematical ideas behind a quadratic relation. The resulting dynamic GeoGebra model serves three main purposes. First, it is the end-product of a problem solving process and can be evaluated to assess a student's understanding of the topic. Second, others can use it as a conceptual model to learn about the problem situation at a different level or as an initial step toward *learning by modeling*. Third, the GeoGebra model can be used as new starting points for higher-order explorations since it can be modified, extended, or incorporated into other instructional units of various mathematical focuses.

Pi

As another design example, we look at the most familiar concept of mathematics, *Pi*, which is defined as the ratio between the circumference of a circle and its diameter. In our experience, few students have trouble recalling the estimated value of *Pi*, but quite a few students cannot describe what it is beyond giving the number *3.14*. There is a long history behind the mathematical idea of *Pi* and many methods for estimating its value (Beckmann, 1976). However, most seem to require advanced mathematical knowledge such as power series or the concept of limit. Our goal is to get students to explore the mathematical idea of *Pi* and build a valid mental model that provides meaning in their future work involving *Pi*.

In light of the ubiquity of circles in the real world and the technological tools provided in GeoGebra, we decided to have students collect data about circles. Specifically, they were asked to find a variety of circles at home, measure them in inches, and record their data in the form of (diameter, circumference) pairs. Except for some measurement errors caused by the ruler, this is a trivial task. It serves as a starting point for further mathematization of the properties of a circle. As a subsequent activity, students were asked to plot these ordered pairs in the GeoGebra environment. Although a single pair of (diameter, circumference) by itself is insignificant, 17 pairs do tend to form a pattern when plotted as shown in Figure 7. This is the first step of *Pi* modeling, where a visual pattern points to the relationship between the circumference and the diameter of a circle.

The next step involves some form of regression analysis, which is beyond the scope of middle-grades mathematics. However, if regression analysis is not the primary objective of instruction, we could take advantage of the technological tools to manage the complexity of the task. Within GeoGebra, we could use the tool *Best Fit Line*, which takes a group of points and generates a line of best fit. With our data, the line of best fit is $y = 3.13x + 0.19$. This best-fit line represents a new type of mathematical model which leads to further discussion about the meaning of the *slope* (3.13) and the interpretation of *y-intercept* (0.19), including the influence of individual points.

A variety of *what-if* and *what-if-not* questions (Brown & Walter, 2005) could be further asked using the dynamic GeoGebra construction. For example, what if someone had made a measurement mistake? What if I drag a point away from the majority of the points? What if we had measured 100 circles of different sizes?

Discussion of these questions and the meaning of slope will eventually help students come to understand *Pi* as a ratio between the circumference and the diameter of a circle and construct a meaningful mental model of a circle, where *Pi* indicates how the circumference changes with the diameter of a circle. Such a dynamic mental model is much more powerful for students to make inferences and predictions about relations involving circles than the narrow conception of *Pi* as a number that is about 3.14 and will in the long run help students appreciate the ideas behind *Pi* and similar linear relations as they move forward to higher levels of mathematics.

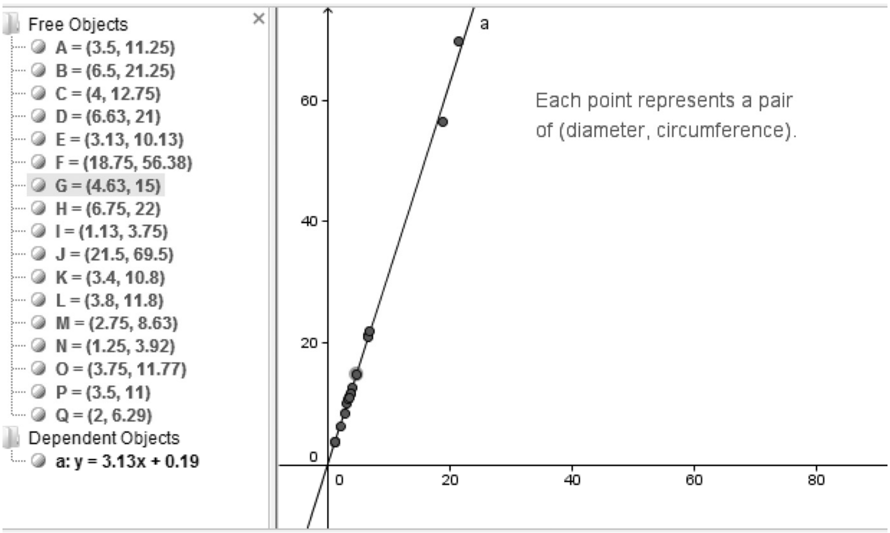

Figure 7. The relationship between the circumference and the diameter of a circle (Point-wise plot and best-fit line).

Similarity

As a third example, we look at the concept of similarity, which is closely related to proportional thinking in the learning processes. Most students have an informal understanding of similarity and can describe it in everyday terms. However, in exploring the concept of similarity in a realistic context, they tend to have difficulty coordinating the multiple quantities involved in a ratio or a proportion. In our work involving GeoGebra, we used the two poles problem as presented below.

> There are two poles erected on the ground (as shown Figure 8). One is six feet tall, and the other is three feet tall. Two ropes are tied from the top of one pole to the bottom of the other, intersecting at point *P*. What is the height of point *P*?

The problem situation should be imaginable to all students. In fact, they could conduct a hands-on or physical experiment and measure the height of Point *P* with respect to the ground. A question that naturally arises with the students is "How far apart are the two poles?" The distance between the two poles is not given in the original problem and this hinders students' attempts to solve the problem. When they are asked to experiment with different distances, however, they would begin to have a tentative finding: Perhaps, it does not matter at all!

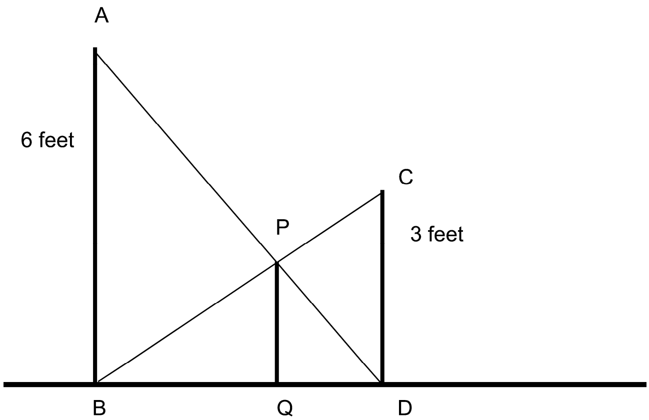

Figure 8. The two-pole problem: What is the height of point P?

While a physical simulation reveals some details of the problem, it has physical limitations: Students can not easily manipulate the problem or extend the problem space. With the physical model as a starting point, students can then move ahead to a GeoGebra simulation, which calls for further mathematization of the problem. In our experience with prospective and in-service teachers, we have repeatedly found that this step is a frustrating and yet motivating stage. When asked to build a GeoGebra model to represent the problem scenario, the vast majority would make a visual model of the original picture using lines and segments without attending to the mathematical aspects and assumptions of the problem. The visual model looks like a GeoGebra-based model of the problem; but when dragged, it collapses to their disappointment. This is a *good* mistake since it reveals the limitations of a visual model and helps students attend to the mathematical aspects of the problem. This shows the diagnostic power of a GeoGebra-based dynamic construction. A brief conversation with the class would quickly lead to the observation that the two poles should be perpendicular to the ground and, indeed, the ground does not have to be horizontal in a mathematical model.

Once students have completed this first step of mathematization from the context to a mathematically valid model, they could move on to the next level and use the GeoGebra model to address the original question about the height of point *P*, which is two feet. Subsequent exploration will unveil the fact that the distance between the two poles is irrelevant. No matter how far apart they are, point *P* is always two feet above the ground, even if the ground is slanted. When the *Trace* feature is turned on for point *P*, we could move the two poles back and forth to collect more evidence in support of the observation, as shown in Figure 9.

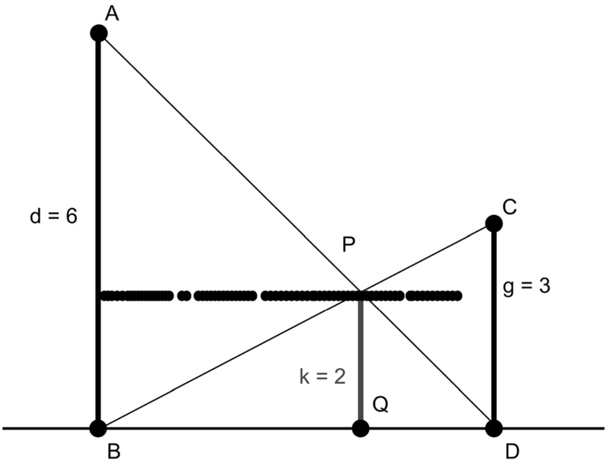

Figure 9. The height of point P stays constant regardless of the distance between B and D.

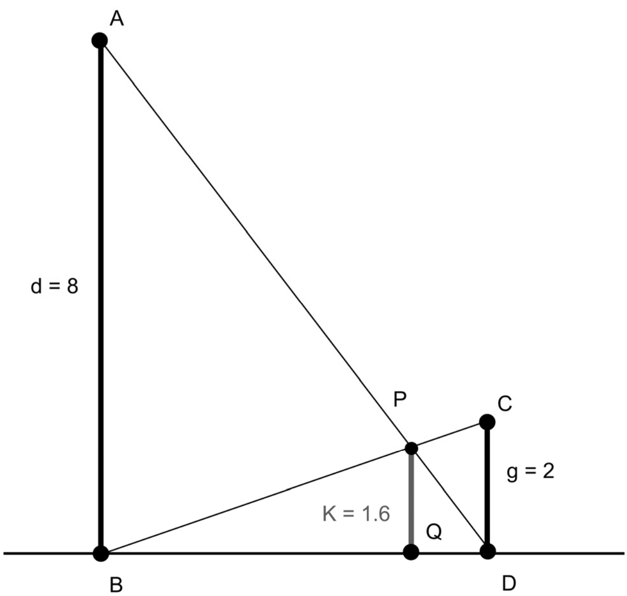

Figure 10. The height of point P is not the ratio between the two poles.

In a dynamic GeoGebra model, the discussion does not end with finding the height of point *P*, which in fact is not the overarching goal of the activity. Students should be prompted for the next level of exploration—how is the height of point *P* related to those of the two poles? Many students might come to a quick hypothesis that the height of point *P* is the length of the longer pole divided by that of the short one, which is indeed true numerically in this specific case. When such a hypothesis is discussed by the whole class, some students would suggest experimenting with the dynamic model, since they could change the initial conditions and generate numerous cases, which would lead to the rejection of their previous hypothesis. Although exploration through GeoGebra-based dynamic modeling does not yield an immediate rule for the relationship between the height of point *P* and the lengths of the two poles, it is a very meaningful and authentic learning process, which may serve as the foundation of a valid mental model for proportional reasoning in that it goes beyond a formula such as $a/b = c/d$ and reveals the structure of the problem scenario.

When students interact with the dynamic GeoGebra model, making and/or rejecting their hypotheses, they may potentially see the invariant relational structure among the numerous cases. There are always two pairs of similar triangles involved, and the two triangles share a common side. Eventually, they may be scaffolded to articulate ratios with the two triangles and derive a rule for the height of point *P*, which is in the form of $(AB \times CD)/(AB+CD)$ and is applicable to all cases. While the rule itself is interesting, it is the holistic experience that will help students appreciate the mathematical ways of reasoning and the rationale behind the rules in mathematics. Along the learning trajectory, GeoGebra plays a variety of cognitive roles. First, it helps students understand the problem and identify gaps in their mathematical knowledge. Second, it helps solve the original problem and open doors for further exploration. Third, it helps students reason with the model, formulating and/or rejecting their hypotheses. Fourth, it serves as a conceptual model for proportional reasoning, which may eventually be incorporated into a student's mental models for future encounters with similar problems. Finally, it shows how tools support and limit our perception of mathematical processes. To summarize, it is the whole experience from physical modeling and GeoGebra modeling, to advanced mathematical reasoning that provides students with a perspective on the complexity of the problem, the human nature of mathematical reasoning, and a paradigmatic case for mathematical problem solving. Indeed, the formula or the recall of such a formula does not illuminate the richness of the relations underlying the problem scenario and its pedagogical values.

In this section, we provided three design examples to showcase the relevance of the theoretical framework discussed previously in our re-conceptualization of school mathematics for the purpose of learning with understanding. There are a variety of mathematical topics that lend themselves to this type of experimentation which strives to provide certain valid mental models for students to make sense of and develop a holistic perspective on the end-products of mathematical investigations—rules and formulas. The scenarios presented above could well be replaced with similar ones, depending on the instructional context, the prior knowledge of the students, and the specific learning objectives. The fundamental principle is that students should be engaged in conceptual modeling in order to develop valid and generative mental models in support of their mathematical learning. They either learn with ready-made models or learn by building or modifying models, using the technological tools available in the learning environment.

CONCLUSIONS

Mathematics is a human activity (Freudenthal, 1973). Mathematics learning and instruction are endowed with all the complexities of such a human endeavor, which range from the multi-dimensionality of the subject matter, the diverse backgrounds of students, to the evolving learning environments, including technological tools and educational goals. Virtually all mathematical ideas, even the very basic ideas in school mathematics, assume their significance with respect to an underlying conceptual system (Lesh, 2006). In this chapter, we first recognized the complexity of mathematics learning and the corresponding call for meaning and understanding in the ongoing mathematics education reforms. While it is relatively easy to identify a lack of understanding such as in the case of rule-based recall of facts, it is challenging to define what understanding is in the context of mathematics learning. A brief literature review further revealed the complexities of understanding in various contexts from learning theories to philosophy. In light of the research and theoretical developments in the past three decades in mathematics education, it seems reasonable to characterize mathematical understanding as a matter of having a world of dynamic mental models that are consistent with the conceptual systems of mathematics and can be called upon in specific situations in support of decision-making and predictions. The complexity of a mathematical idea, especially its connections to an underlying system, further requires the use of multiple representations and their dynamic interconnections. Given the internal nature of mental models, conceptual modeling becomes a necessary mediator to foster changes and developments in a learner’s mental world. In exploratory modeling, learners interact with ready-made systems as a way to learn about the underlying structure of the system; in expressive modeling, learners construct or modify models as a way to externalize, reflect on, and modify their mental models (Doerr & Pratt, 2008). At a higher level, as learners solve problems in a model-centered environment, they further construct a mental model of themselves as problem solvers, which includes their beliefs, attitudes, and identity in relation to mathematics learning (Goldin, 2007; Norman, 1983).

The dynamic nature of mathematical understanding and the corresponding needs for multiple representations serve as a theoretical foundation for the integration of technological tools such as GeoGebra, which provides the utilities for learners to construct mathematical models. In a traditional setting, an expert’s dynamic understanding of a mathematical idea is usually hidden from the observers. To some extent, GeoGebra models help experts better externalize their mental models of mathematics for the purpose of personal reflection, and more importantly, as conceptual systems to facilitate novices’ learning.

From a design perspective, we synthesized the basic principles of Realistic Mathematics Education (RME) and Model-Facilitated Learning (MFL), which, though developed in different fields, share a common theoretical orientation—facilitating the learning of complex subject matter in a meaningful manner using models and modeling as a pedagogical tool to manage complexity and promote increasingly higher levels of understanding. While RME is deeply rooted in the search for meaning in the past three decades of mathematics education research and development, MFL has a solid foundation in learning and instructional design theories, with a strong commitment to the integration of new interactive technologies. We further considered the theory of Instrumental Genesis (IG) as a theoretical lens to examine the use of tools in mathematics learning. As learners make use of new technological resources such as GeoGebra in mathematical problem solving, their mathematical conceptions or mental models may become increasingly instrumented entities. In our work with prospective and in-service teachers, we have collected data in support of this theoretical construct. For example, when asked to find the area of a triangle whose vertices are given in terms of coordinates, many teachers tended to plot the three points, define a polygon, and then read its area from the GeoGebra environment, even if the three vertices were special cases and the area required only simple computations. Once tools become part of their mental resources, they seem to pose challenges to the traditional conceptions of mathematics and especially assessments. In short, RME, MFL, and IG provide a unified theoretical framework for us to examine the design and learning processes in our GeoGebra-integrated mathematics courses and professional development projects. As we gather more empirical data from the teachers and their students on a variety of mathematical topics in a variety of settings, we may need to further refine our theoretical constructs and clarify the relevance and limitations of the basic principles.

As design examples, we presented our preliminary work on the quadratic relations, *Pi*, and similarity in GeoGebra-integrated mathematics courses and professional development, which demonstrate how typical ideas in school mathematics should and could be reconceptualized, recontextualized, and problematized for the purposes of meaningful learning. One common characteristic of the three examples is our effort to engage students in whole-task explorations while providing just-in-time support with regard to component skills (van Merriënboer, Clark, & de Crook, 2002; van Merriënboer & Kirschner, 2007). Basic skills, such as plotting points and constructing perpendicular lines, are meaningful mostly because of their connections to the whole task and they are scaffolded on demand as part of the whole task.

While GeoGebra trivializes a host of traditional mathematical tasks such as graphing functions, solving equations, and finding geometric reflections, it does open the door for much more interesting and motivating scenarios of mathematical explorations and provides a platform for designing and implementing inquiry-based learning (Barron & Darling-Hammond, 2008). Technology changes what mathematics can be investigated with students and how traditional mathematical ideas should be taught (NCTM, 2000). As new technologies enter the lives of students at school and beyond, traditional problem solving could be further considered from a modeling and models perspective, incorporating the evolving needs of students and expectations of society (Lesh & Doerr, 2003). Furthermore, in light of the versatile nature of GeoGebra and its ongoing development, GeoGebra lends itself to a variety of theoretical frameworks for mathematics education. Just as we recognize the dynamic nature of mathematical understanding and the use of GeoGebra, we seek to embrace a dynamic and diverse understanding of instructional and learning theories (Jonassen, 2005) as the world community joins hands in charting out the challenges and opportunities of quality mathematics education for all.

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Lingguo Bu
Department of Curriculum and Instruction
Southern Illinois University Carbondale, USA

J. Michael Spector
Learning and Performance Support Laboratory (LPSL)
University of Georgia, USA

Erhan Selcuk Haciomeroglu
School of Teaching, Learning, and Leadership
University of Central Florida, USA